A matrix model for bilayered quantum Hall systems

2004 J. Phys. A: Math. Gen. 37 3147

(http://iopscience.iop.org/0305-4470/37/9/007)

J. Phys. A: Math. Gen. **37** (2004) 3147–3157 PII: S0305-4470(04)63759-0

A matrix model for bilayered quantum Hall systems

Ahmed Jellal[1] **and Michael Schreiber**

Institut für Physik, Technische Universität, D-09107 Chemnitz, Germany

E-mail: jellal@gursey.gov.tr

Received 21 May 2003, in final form 2 December 2003
Published 18 February 2004
Online at stacks.iop.org/JPhysA/37/3147 (DOI: 10.1088/0305-4470/37/9/007)

Abstract
We develop a matrix model to describe bilayered quantum Hall fluids for a series of filling factors. Considering two coupling layers, and starting from a corresponding action, we construct its vacuum configuration at $\nu = q_i K_{ij}^{-1} q_j$, where K_{ij} is a 2×2 matrix and q_i is a vector. Our model allows us to reproduce several well-known wavefunctions. We show that the wavefunction $\Psi_{(m,m,n)}$ constructed years ago by Yoshioka, MacDonald and Girvin for the fractional quantum Hall effect at filling factor $\frac{2}{m+n}$ and in particular $\Psi_{(3,3,1)}$ at filling $\frac{1}{2}$ can be obtained from our vacuum configuration. The unpolarized Halperin wavefunction and especially that for the fractional quantum Hall state at filling factor $\frac{2}{5}$ can also be recovered from our approach. Generalization to more than two layers is straightforward.

PACS number: 74.43.Cd

1. Introduction

The quantum Hall (QH) effect has bred many interesting theories. Indeed, Laughlin's wavefunctions [1] are good wavefunctions for describing the fractional quantum Hall effect (FQHE)[2] at filling factor $\nu = \frac{1}{m}$, where m is an odd integer. For other filling factors several attempts have been suggested to extend Laughlin's theory by adopting different approaches and assumptions. In particular, Halperin [3] proposed a family of generalized Laughlin wavefunctions that could incorporate reversed spins. In fact a candidate for an unpolarized wavefunction at filling factor $\frac{2}{5}$ was given. Subsequently, Yoshioka *et al* [4] generalized the Laughlin wavefunctions to those of the bilayered QH systems and derived those corresponding to the $\nu = \frac{1}{2}$ state. Moreover, other theories have been elaborated and have led to the understanding of the observed values of ν, in particular $\nu = \frac{5}{2}$ [5] as well as others [6].

The first experimental indications of an unpolarized ground-state spin configuration in the FQHE came with the discovery of the $\nu = \frac{5}{2}$ state [7] and later the $\nu = \frac{4}{3}$ state [8].

[1] Permanent address: Physics Department, Faculty of Sciences, Chouaib Doukkali University, Road Ibn Mâachou PO Box 20, El Jadida, Morocco.

More compelling evidence for novel spin phenomena in the FQHE was subsequently reported [9]. On the other hand, it was shown experimentally that multi-layer systems also exhibit the FQHE [10]. In fact, several filling factors have been observed, for instance the $\nu = \frac{1}{2}$ state [11] and $\nu = \frac{9}{2}, \frac{11}{2}, \ldots$ [12].

Recently, Susskind [13] proposed a novel method of investigating the FQHE. He claimed that the non-commutative Chern–Simons theory (NCCS) at level k is exactly equivalent to Laughlin's theory at the filling factor $\nu_{\mathrm{S}} = \frac{1}{k}$. He formulated his approach as a matrix theory similar to that describing $D0$-branes in string theory. However, Susskind's theory is an alternative approach to the FQHE, which so far has not produced anything new but has just recovered the Laughlin approach by adopting a new formalism. Nevertheless it remains a new way of thinking and is worth studying in the hope that it will bring new results in the future.

Although the proposed matrix model seems to reproduce the basic features of the Laughlin QH droplets, still some problems remain to be solved. Indeed, Susskind's approach is valid only for infinite matrices and also shows an anomaly for $k = 0$. To solve these problems, Polychronakos [14] introduced a boundary term to the Susskind matrix model. He proposed a finite matrix model as a regularized version of the NCCS theory. It allowed him to find a quantum correction to ν_{S}, where k is shifted to $k + 1$ and the filling factor became $\nu_{\mathrm{PS}} = \frac{1}{k+1}$. As another consequence, he pointed out that his matrix model is equivalent to the Calogero model [15].

Sometimes later, observing that the Laughlin wavefunctions can be mapped onto many-body wavefunctions of the harmonic oscillator, Hellerman and Van Raamsdonk [16] built a complete minimal basis of wavefunctions of the theory at an arbitrary level k and rank M, see also [17]. Other investigations about the relation between NCCS and Laughlin fluids can be found in [18, 19]. Subsequently, the Susskind model and its regularized version introduced by Polychronakos were extended to FQH states that are not of Laughlin type: a multicomponent Chern–Simons approach was introduced [20] and another proposal based on the Haldane hierarchy [21] was developed [22, 23].

Despite the progress in the study of the FQH fluids in the framework of NCCS matrix model, several open questions remain which have not been addressed so far. One of these questions concerns the wavefunctions that are not of Laughlin type. In fact there are many wavefunctions that have been constructed years ago, e.g. by Yoshioka *et al*, Halperin and others, but cannot be recovered by what is developed so far.

In what follows we propose a matrix model to investigate the possibility of obtaining two of those wavefunctions. This can be done by extending the Susskind–Polychronakos model to deal with the QH fluids at the filling factor [24]

$$\nu = q_i K_{ij}^{-1} q_j \tag{1}$$

where K_{ij} is an $N \times N$ matrix and q_i is a vector. The basic idea is to consider several Susskind–Polychronakos systems, let us say M systems, with an interaction between them and suppose that all systems possess the same number of particles. In the QHE language, this picture is equivalent to considering multi-layered systems. Without loss of generality, we fix $M = 2$, but as we will see later our analysis can directly be extended to the generic case $M \geqslant 3$.

We start by writing down an appropriate action as a sum of two terms of the free and the interacting parts. Subsequently, we derive the corresponding Hamiltonian, which of course contains an interaction. Using a unitary transformation, we show that this Hamiltonian can be transformed to a diagonalized one. Next, we determine the vacuum configuration that allows us to recover two different states. Indeed, we show that how the Yoshioka–MacDonald–Girvin wavefunctions at the filling factor $\nu = \frac{2}{m+n}$ can be obtained from our model and in particular that describing the FQHE at $\nu = \frac{1}{2}$. Moreover, the unpolarized Halperin wavefunctions will be derived and especially those corresponding to the $\nu = \frac{2}{5}$ state.

In section 2 we recall briefly the NCCS matrix model description of the Laughlin fluid. In section 3, we propose an action describing a system of two layers, we derive the Gauss law constraint as well as the equations of motion for the different variables. A quantum mechanical analysis will be the subject of the next section, where we develop a Hamiltonian that corresponds to the system under consideration. Under rotation, we define a set of matrices of harmonic-oscillator operators to diagonalize the system. In section 5, we build the vacuum configuration that satisfies the constraint. A link with literature will be discussed in the last section where the two wavefunctions mentioned above will be recovered. We conclude our paper by raising some questions to be investigated in the forthcoming works.

2. Chern–Simons matrix model

Starting from the matrix formulation of a two-dimensional system with a large number of electrons in the presence of a perpendicular magnetic field B, Susskind [13] showed that the resulting effective theory is a non-commutative $U(1)$ Chern–Simons gauge theory at level $k = B\theta$. As a consequence, he found a relation

$$\rho = \frac{1}{2\pi\theta} \tag{2}$$

which links the non-commutative parameter θ to the density of electrons ρ. By using the definition of the filling factor

$$\nu = \frac{2\pi\rho}{B} \tag{3}$$

in the system of units $(\hbar, e, c)$, it is easily seen that the fraction ν can be written in terms of the parameter θ as

$$\nu = \frac{1}{B\theta}. \tag{4}$$

This beautiful relation is one of the interesting results obtained recently by Susskind in dealing with the FQH fluids.

Moreover, by exploring the possibility of developing a consistent finite matrix model for the description of the FQH droplet, Polychronakos [14] suggested to include a new field into the Susskind model. The proposed action is given by

$$S = \int \mathrm{d}t\, \frac{B}{2} \operatorname{Tr}\left\{\epsilon^{ab}(\dot{X}_a + \mathrm{i}[A_0, X_a])X_b + 2\theta A_0 - \omega X_a^2\right\} + \psi^\dagger(\mathrm{i}\dot{\psi} - A_0\psi) \tag{5}$$

where $X_a, a = 1, 2$ are $N \times N$ matrices and ψ is a complex N-vector, and $\epsilon^{12} = -\epsilon^{21} = 1$, $\epsilon^{aa} = 0$. The action is invariant under the gauge group $U(N)$ and the matrix model variables transform as

$$X_a \to U X_a U^{-1} \qquad \psi \to U\psi. \tag{6}$$

The equation of motion for A_0 leads to the Gauss law constraint

$$G \equiv -\mathrm{i}B[X_1, X_2] + \psi\psi^\dagger - B\theta = 0. \tag{7}$$

The trace of this equation gives

$$\psi^\dagger\psi = NB\theta. \tag{8}$$

Upon quantization, the matrix elements of X_a and the components of ψ become operators, obeying the commutation relations

$$\left[\psi_i, \psi_j^\dagger\right] = \delta_{ij} \qquad [(X_1)_{ij}, (X_2)_{kl}] = \frac{\mathrm{i}}{B}\delta_{il}\delta_{jk}. \tag{9}$$

The Hamiltonian can be obtained from (5) as

$$H = \omega\left(\frac{N^2}{2} + \sum A_{nm}^\dagger A_{mn}\right) \tag{10}$$

where the $N \times N$ matrix of harmonic-oscillator operators is defined by

$$A_{nm} = \sqrt{\frac{B}{2}}\,(X_1 + \mathrm{i}X_2)_{nm}. \tag{11}$$

The corresponding wavefunction is [16]

$$|k\rangle = \left[\epsilon^{i_1\ldots i_N}\psi_{i_1}^\dagger(\psi^\dagger A^\dagger)_{i_2}\cdots(\psi^\dagger A^{\dagger N-1})_{i_N}\right]^k|0\rangle \tag{12}$$

where the vacuum $|0\rangle$ is annihilated by A and ψ and ϵ is the fully antisymmetric tensor. This is a physical state and therefore satisfies the relation

$$G|k\rangle = 0. \tag{13}$$

It is similar to the Laughlin wavefunction [1] at the filling factor

$$\nu = \frac{1}{k+1}. \tag{14}$$

Subsequently, one of us and others [22, 23] generalized the above results to any filling factor which can be expressed as

$$\nu_{k_1k_2} = \frac{1}{k_1} + \frac{1}{k_2} \tag{15}$$

and in particular to level two of the Haldane hierarchy [21]

$$\nu_{p_1p_2} = \frac{p_2}{p_1p_2 - 1} \tag{16}$$

by setting

$$k_1 = p_1 \qquad k_2 = p_1(p_1p_2 - 1) \tag{17}$$

where p_1 is an odd and p_2 is an even integer.

3. Two coupling matrices model

We consider two systems with a total number of particles $M_1 + M_2$ which interact with each other. Such systems can be seen like two coupling layers i containing M_i particles. The appropriate action to describe the FQH fluids of the whole system at filling factor (1) is given by

$$S = \int \mathrm{d}t \sum_j \frac{K_{jj}}{2\theta}\,\mathrm{Tr}\left\{\epsilon^{ab}\left(\dot{X}_a^{(j)} + \mathrm{i}\left[A_0, X_a^{(j)}\right]\right)X_b^{(j)} + 2\theta A_0 - \omega_j\left(X_a^{(j)}\right)^2\right\} \\ + \psi^{(j)\dagger}(\mathrm{i}\dot{\psi}^{(j)} - A_0\psi^{(j)}) + \int \mathrm{d}t\, K_{12}\left\{\frac{\omega_{12}}{\theta}\,\mathrm{Tr}\left(X_a^{(1)}X_a^{(2)}\right) + \psi^{(1)}\psi^{(2)}\right\} \tag{18}$$

which involves two copies of the single-layer action (5) forming the free part. It also contains an interacting part, where the scalar K_{12} plays the role of a coupling parameter between the layers 1 and 2. The ratio $\frac{K_{jj}}{\theta}$ is basically the magnetic field B.

It is clear that for $K_{12} = 0$, the total system becomes decoupling. Note that as far as the total action is concerned, the full gauge symmetry is $U(M_1) \times U(M_2)$. The matrix model variables transform under this invariance as

$$X_a^{(i)} \to U X_a^{(i)} U^{-1} \qquad \psi^{(i)} \to U\psi^{(i)}. \tag{19}$$

Compared to the original matrix model, there is the potential term

$$V = \sum_j \frac{K_{jj}}{2\theta}\omega_j \operatorname{Tr}\left(X_a^{(j)}\right)^2 - \frac{K_{12}}{\theta}\omega_{12} \operatorname{Tr}\left(X_a^{(1)} X_a^{(2)}\right) \tag{20}$$

analogous to the potential of two coupled harmonic oscillators [25] in the two-dimensional space. This provides a Hamiltonian for the theory.

The Gauss law constraint can be obtained by evaluating the equation of motion for A_0. In our case it reads

$$\mathcal{G} \equiv -\mathrm{i}K_{11}\left[X_1^{(1)}, X_2^{(1)}\right] - \mathrm{i}K_{22}\left[X_1^{(2)}, X_2^{(2)}\right] + (\psi^{(1)}\psi^{(1)\dagger} + \psi^{(2)}\psi^{(2)\dagger} - K_{11} - K_{22}) = 0 \tag{21}$$

where its trace gives

$$\psi^{(1)\dagger}\psi^{(1)} + \psi^{(2)\dagger}\psi^{(2)} = M_1 K_{11} + M_2 K_{22}. \tag{22}$$

Other equations of motion can also be calculated. For the X we get

$$\begin{aligned} &K_{11}\epsilon^{ab}\dot{X}_a^{(1)} + K_{11}\omega_1 X_a^{(1)} + K_{12}\omega_{12} X_a^{(2)} = 0 \\ &K_{22}\epsilon^{ab}\dot{X}_a^{(2)} + K_{22}\omega_2 X_a^{(2)} + K_{12}\omega_{12} X_a^{(1)} = 0 \end{aligned} \tag{23}$$

while for the ψ we obtain

$$\mathrm{i}\psi^{(1)\dagger} + K_{12}\psi^{(2)} = 0 \qquad \mathrm{i}\psi^{(2)\dagger} + K_{12}\psi^{(1)} = 0. \tag{24}$$

Of course the last set of equations shows a difference with respect to the decoupled case. It can be solved by using a unitary transformation.

4. Hamiltonian formalism

Let us now consider the proposed model quantum mechanically. We proceed by determining the total Hamiltonian, which describes the system under consideration. It can be obtained from the relation

$$\mathcal{H} = \dot{X}\frac{\partial L}{\partial \dot{X}} - L \tag{25}$$

where $\frac{\partial L}{\partial \dot{X}}$ defines the conjugate momentum. This leads to a Hamiltonian as the sum of the free and the interacting parts as

$$\mathcal{H} = \sum_j \frac{K_{jj}}{2\theta}\omega_j \operatorname{Tr}\left(X_a^{(j)}\right)^2 - \frac{K_{12}}{\theta}\omega_{12} \operatorname{Tr}\left(X_a^{(1)} X_a^{(2)}\right) \tag{26}$$

which is nothing but the confining potential (20). This means that the kinetic energy is negligible compared to V.

It is clear that this form of $\mathcal{H}$ cannot be diagonalized directly. Nevertheless, $\mathcal{H}$ can be transformed to another factorizing Hamiltonian $\mathcal{H}'$. Probably the best way to do this is to perform a rotation by a mixing angle φ of the X to new matrices

$$Y_a^{(1)} = X_a^{(1)}\cos\frac{\varphi}{2} - X_a^{(2)}\sin\frac{\varphi}{2} \qquad Y_a^{(2)} = X_a^{(1)}\sin\frac{\varphi}{2} + X_a^{(2)}\cos\frac{\varphi}{2}. \tag{27}$$

It can easily be checked that this rotation is a unitary transformation. Inserting (27) into (26), one can show that $\mathcal{H}$ transform to

$$\mathcal{H}' = \alpha\,\mathrm{Tr}\left(Y_a^{(1)}\right)^2 + \beta\,\mathrm{Tr}\left(Y_a^{(2)}\right)^2 \tag{28}$$

if the rotating angle satisfies the relation

$$\tan\varphi = \frac{K_{12}\omega_{12}}{K_{11}\omega_1 - K_{22}\omega_2}. \tag{29}$$

The parameters α and β are given by

$$\begin{aligned}\alpha &= \frac{1}{\theta}\left(K_{11}\omega_1\cos^2\frac{\varphi}{2} + K_{22}\omega_2\sin^2\frac{\varphi}{2} - \frac{1}{2}K_{12}\omega_{12}\sin\varphi\right)\\ \beta &= \frac{1}{\theta}\left(K_{11}\omega_1\sin^2\frac{\varphi}{2} + K_{22}\omega_2\cos^2\frac{\varphi}{2} + \frac{1}{2}K_{12}\omega_{12}\sin\varphi\right).\end{aligned} \tag{30}$$

To diagonalize $\mathcal{H}'$, we define two couples of creation and annihilation matrices of harmonic-oscillator operators

$$C_{nm}^{(1)} = \sqrt{\frac{\alpha}{2}}\left(Y_1^{(1)} + \mathrm{i}Y_2^{(1)}\right)_{nm} \qquad C_{nm}^{(2)} = \sqrt{\frac{\beta}{2}}\left(Y_1^{(2)} + \mathrm{i}Y_2^{(2)}\right)_{nm}. \tag{31}$$

They satisfy the commutation relations

$$\left[C_{nm}^{(1)}, C_{n'm'}^{(1)\dagger}\right] = \delta_{nm'}\delta_{n'm} \qquad \left[C_{ij}^{(2)}, C_{i'j'}^{(2)\dagger}\right] = \delta_{ij'}\delta_{i'j} \tag{32}$$

while all other commutators vanish. Now $\mathcal{H}'$ can be rewritten as

$$\mathcal{H}' = \frac{\alpha}{2}\left(2\mathcal{M}_1 + M_1^2\right) + \frac{\beta}{2}\left(2\mathcal{M}_2 + M_2^2\right) \tag{33}$$

where the number operators

$$\mathcal{M}_1 = \sum_{n,m=1}^{M_1} C_{mn}^{(1)\dagger}C_{nm}^{(1)} \qquad \mathcal{M}_2 = \sum_{i,j=1}^{M_2} C_{ij}^{(2)\dagger}C_{ji}^{(2)} \tag{34}$$

are counting the M_1 and M_2 particles. Thus under the unitary transformation the system becomes decoupling.

5. Ground-state wavefunctions

To begin we emphasize a difference between the ground state of two coupled harmonic oscillators in terms of the coordinates x_i and that in terms of their mapped representations y_i. The wavefunction

$$\psi_0(\vec{y}) \sim \exp\left\{-\alpha y_1^2 - \beta y_2^2\right\} \tag{35}$$

is separable in the variables y_1 and y_2. However, for the variables x_1 and x_2, the wavefunction (35) reads

$$\psi_0(\vec{x}) \sim \exp\left\{-\alpha\left(x_1\cos\frac{\varphi}{2} - x_2\sin\frac{\varphi}{2}\right)^2 - \beta\left(x_1\sin\frac{\varphi}{2} + x_2\cos\frac{\varphi}{2}\right)^2\right\}. \tag{36}$$

Next, we will see how these ground states can be extended to the matrix model formalism. We begin to determine that for the matrices Y. By transforming the Gauss law constraint to the variables Y, i.e.

$$\begin{aligned}&\left(K_{11}\cos^2\frac{\varphi}{2} + K_{22}\sin^2\frac{\varphi}{2}\right)\left[Y_1^{(1)}, Y_2^{(1)}\right] + \left(K_{11}\sin^2\frac{\varphi}{2} + K_{22}\cos^2\frac{\varphi}{2}\right)\left[Y_1^{(2)}, Y_2^{(2)}\right]\\ &\qquad + \frac{1}{2}(K_{11} - K_{22})\sin\varphi\left\{\left[Y_1^{(1)}, Y_2^{(2)}\right] + \left[Y_1^{(2)}, Y_2^{(1)}\right]\right\}\\ &\quad = \mathrm{i}\,\theta\left(K_{11} + K_{22} - \phi^{(1)}\phi^{(1)\dagger} - \phi^{(2)}\phi^{(2)\dagger}\right)\end{aligned} \tag{37}$$

where the Polychronakos fields are also rotated to new fields

$$\phi^{(1)} = \psi^{(1)} \cos\frac{\varphi}{2} - \psi^{(2)} \sin\frac{\varphi}{2} \qquad \phi^{(2)} = \psi^{(1)} \sin\frac{\varphi}{2} + \psi^{(2)} \cos\frac{\varphi}{2}. \tag{38}$$

For simplicity let us fix $K_{11} = K_{22} = K$, then (37) becomes

$$\left[Y_1^{(1)}, Y_2^{(1)}\right] + \left[Y_1^{(2)}, Y_2^{(2)}\right] = 2\,\mathrm{i}K\theta\left(1 - \frac{1}{2K}\phi^{(1)}\phi^{(1)\dagger} - \frac{1}{2K}\phi^{(2)}\phi^{(2)\dagger}\right). \tag{39}$$

Now it is clear that the ground state is simply a tensor product between those states corresponding to each layer

$$\begin{aligned}|K\rangle = &\left[\epsilon^{i_1\cdots i_{M_1}}\phi_{i_1}^{(1)\dagger}(\phi^{\dagger(1)}C^{(1)\dagger})_{i_2}\cdots(\phi^{(1)\dagger}C^{(1)\dagger M_1-1})_{i_{M_1}}\right]^K \\ &\left[\epsilon^{j_1\cdots j_{M_2}}\phi_{j_1}^{(2)\dagger}(\phi^{(2)\dagger}C^{(2)\dagger})_{j_2}\cdots(\phi^{(2)\dagger}C^{(2)\dagger M_2-1})_{j_{M_2}}\right]^K|0\rangle.\end{aligned} \tag{40}$$

The ground state (40) can be mapped in terms of the operators of the matrices X by expressing the matrices C of harmonic-oscillator operators in terms of those corresponding to the matrices X. Using (27), one can show that (31) takes the form

$$C_{nm}^{(1)} = \sqrt{\frac{\alpha}{B}}\left(A^{(1)}\cos\frac{\varphi}{2} - A^{(2)}\sin\frac{\varphi}{2}\right)_{nm} \qquad C_{nm}^{(2)} = \sqrt{\frac{\beta}{B}}\left(A^{(1)}\sin\frac{\varphi}{2} + A^{(2)}\cos\frac{\varphi}{2}\right)_{nm} \tag{41}$$

where the operators

$$A_{nm}^{(1)} = \sqrt{\frac{B}{2}}\left(X_1^{(1)} + \mathrm{i}X_2^{(1)}\right)_{nm} \qquad A_{nm}^{(2)} = \sqrt{\frac{B}{2}}\left(X_1^{(2)} + \mathrm{i}X_2^{(2)}\right)_{nm} \tag{42}$$

commute:

$$\left[A_{nm}^{(1)}, A_{n'm'}^{(1)\dagger}\right] = \delta_{nm'}\delta_{n'm} \qquad \left[A_{ij}^{(2)}, A_{i'j'}^{(2)\dagger}\right] = \delta_{ij'}\delta_{i'j}. \tag{43}$$

Inserting (38) and (41) in (40), we obtain

$$\begin{aligned}|K\rangle = &\Bigg[\epsilon^{i_1\ldots i_{M_1}}\left(\psi^{(1)\dagger}\cos\frac{\varphi}{2} - \psi^{(2)\dagger}\sin\frac{\varphi}{2}\right)_{i_1}\cdots \\ &\quad\times\left\{\left(\psi^{(1)\dagger}\cos\frac{\varphi}{2} - \psi^{(2)\dagger}\sin\frac{\varphi}{2}\right)\left(A^{(1)\dagger}\cos\frac{\varphi}{2} - A^{(2)\dagger}\sin\frac{\varphi}{2}\right)^{M_1-1}\right\}_{i_{M_1}}\Bigg]^K \\ &\Bigg[\epsilon^{j_1\ldots j_{M_2}}\left(\psi^{(1)\dagger}\sin\frac{\varphi}{2} + \psi^{(2)\dagger}\cos\frac{\varphi}{2}\right)_{j_1}\cdots \\ &\quad\times\left\{\left(\psi^{(1)\dagger}\sin\frac{\varphi}{2} + \psi^{(2)\dagger}\cos\frac{\varphi}{2}\right)\left(A^{(1)\dagger}\sin\frac{\varphi}{2} + A^{(2)\dagger}\cos\frac{\varphi}{2}\right)^{M_2-1}\right\}_{j_{M_2}}\Bigg]^K|0\rangle.\end{aligned} \tag{44}$$

In what follows, we proceed without the use of the unitary transformation to construct the wavefunction $|\Phi\rangle$ describing the system of $M_1 + M_2$ electrons at filling factor (1). One has to realize a physical state $|\Phi\rangle$ that satisfies the Gauss law constraint (21)

$$\mathcal{G}|\Phi\rangle = 0 \tag{45}$$

and allows us to establish a link with two well-known wavefunctions. May be the best way to do this is to define two operators

$$A = A^{(1)} \otimes A^{(2)} \qquad \psi = \psi^{(1)} \otimes \psi^{(2)} \tag{46}$$

where $\otimes$ is the tensor product. Using these matrices of harmonic-oscillator operators, we build a vacuum configuration

$$|\Psi\rangle = \left[\epsilon^{i_1\cdots i_{M_1}} \psi_{i_1}^{\dagger(1)} (\psi^{(1)\dagger} A^{(1)\dagger})_{i_2} \cdots (\psi^{(1)\dagger} A^{(1)\dagger M_1-1})_{i_{M_1}}\right]^{K_{11}-K_{12}}$$
$$\left[\epsilon^{j_1\cdots j_{M_2}} \psi_{j_1}^{(2)\dagger} (\psi^{(2)\dagger} A^{(2)\dagger})_{j_2} \cdots (\psi^{(2)\dagger} A^{(2)\dagger M_2-1})_{j_{M_2}}\right]^{K_{22}-K_{12}}$$
$$\left[\epsilon^{k_1\cdots k_{M_1+M_2}} \psi_{k_1}^{\dagger} (\psi^{\dagger} A^{\dagger})_{k_2} \cdots (\psi^{\dagger} A^{\dagger M_1+M_2-1})_{k_{M_1+M_2}}\right]^{K_{12}} |0\rangle. \tag{47}$$

which satisfies the Gauss law constraint (21) and therefore we have

$$(\psi^{(1)}\psi^{(1)\dagger} + \psi^{(2)}\psi^{(2)\dagger} - M_1 K_{11} - M_2 K_{22})|\Phi\rangle = 0. \tag{48}$$

Novel about this vacuum configuration is that one can interpret the term

$$\left[\epsilon^{k_1\cdots k_{M_1+M_2}} \psi_{k_1}^{\dagger} (\psi^{\dagger} A^{\dagger})_{k_2} \cdots (\psi^{\dagger} A^{\dagger M_1+M_2-1})_{k_{M_1+M_2}}\right]^{K_{12}} \tag{49}$$

as an inter-layer correlation. In conclusion, our configuration could well be a good ansatz for the ground states of double-layered FQH fluids in the formalism of the NCCS matrix model. This will be clarified in the next section.

6. Link with literature

Here we show how the Yoshioka–MacDonald–Girvin and Halperin wavefunctions describing respectively the double-layer and the unpolarized QH systems can be recovered from our vacuum configuration (47).

Before starting, we note that for any N-dimensional vector $\psi^\dagger$ and $N \times N$ matrix $A^\dagger$, the expression of the form

$$F(\psi^\dagger, A^\dagger) = \epsilon^{i_1\cdots i_N} \psi_{i_1}^{\dagger(1)} (\psi^{(1)\dagger} A^{(1)\dagger})_{i_2} \cdots (\psi^{(1)\dagger} A^{(1)\dagger N-1})_{i_N} \tag{50}$$

has a one-to-one correspondence with the polynomial

$$f(z) = \epsilon^{i_1\cdots i_N} z_{i_1}^0 \cdots z_{i_N}^{N-1}. \tag{51}$$

Now our task can be done by defining a new complex variable

$$\zeta_i = \begin{cases} z_i^{(1)} & \text{for} \quad i = 1, \ldots, N \\ z_{i-N}^{(2)} & \text{for} \quad i = N+1, \ldots, 2N \end{cases} \tag{52}$$

assuming that the particle numbers are equal, $M_1 = M_2 = N$, and recalling the Vandermonde determinant

$$\prod_{i<j} (z_i - z_j) = \det\left(z_i^{N-j}\right) = \epsilon^{i_1\cdots i_N} z_{i_1}^0 \cdots z_{i_N}^{N-1}. \tag{53}$$

In terms of the complex coordinates, (47) reads

$$\Psi_{(K_{11},K_{22},K_{12})} = \left[\epsilon^{i_1\cdots i_N} \left(z_{i_1}^{(1)}\right)^0 \cdots \left(z_{i_N}^{(1)}\right)^{N-1}\right]^{K_{11}-K_{12}}$$
$$\left[\epsilon^{j_1\cdots j_N} \left(z_{j_1}^{(2)}\right)^0 \cdots \left(z_{j_N}^{(2)}\right)^{N-1}\right]^{K_{22}-K_{12}}$$
$$\left[\epsilon^{k_1\cdots k_{2N}} \zeta_{k_1}^0 \cdots \zeta_{k_{2N}}^{2N-1}\right]^{K_{12}} \Psi_0. \tag{54}$$

It can be written in standard form as

$$\Psi_{(K_{11},K_{22},K_{12})} = \prod_{i<j} \left(z_i^{(1)} - z_j^{(1)}\right)^{K_{11}} \prod_{i<j} \left(z_i^{(2)} - z_j^{(2)}\right)^{K_{22}} \prod_{i,j} \left(z_i^{(1)} - z_j^{(2)}\right)^{K_{12}} \Psi_0 \tag{55}$$

and now the inter-layer correlation is

$$\prod_{i,j}\left(z_i^{(1)}-z_j^{(2)}\right)^{K_{12}}. \tag{56}$$

Next, we will give two different applications of (55).

6.1. YMG wavefunctions

Considering the two layers and treating them as additional degrees of freedom, the $\nu=\frac{1}{2}$ state was predicted by Yoshioka *et al* [4]. They made a straightforward generalization of the Laughlin wavefunctions to those with the filling factor

$$\nu=\frac{2}{m+n} \tag{57}$$

where m and n are integers. This can be obtained from our analysis by taking

$$K=\begin{pmatrix} m & n\\ n & m\end{pmatrix} \qquad q=(1-1) \tag{58}$$

leading to the wavefunction

$$\Psi_{(m,m,n)}=\prod_{i<j}\left(z_i^{(1)}-z_j^{(1)}\right)^m\prod_{i<j}\left(z_i^{(2)}-z_j^{(2)}\right)^m\prod_{i,j}\left(z_i^{(1)}-z_j^{(2)}\right)^n\Psi_0. \tag{59}$$

Choosing $m=3$ and $n=1$, we recover the FQHE $\nu=\frac{1}{2}$ state corresponding to

$$\Psi_{(3,3,1)}=\prod_{i<j}\left(z_i^{(1)}-z_j^{(1)}\right)^3\prod_{i<j}\left(z_i^{(2)}-z_j^{(2)}\right)^3\prod_{i,j}\left(z_i^{(1)}-z_j^{(2)}\right)\Psi_0. \tag{60}$$

6.2. Halperin wavefunctions

Another interesting result can be obtained. In the Halperin picture [3] in the context of single-layered unpolarized QH systems, the labels 1 and 2 can be considered as an analogue of spin. Following this idea, our bilayered system can be seen as mixing layers of particles with spin up and spin down.

As a consequence, we obtain for $m=3$ and $n=2$ the unpolarized Halperin wavefunction with the filling factor $\frac{2}{5}$ as

$$\Psi_{(3,3,2)}=\prod_{i<j}\left(z_i^{(1)}-z_j^{(1)}\right)^3\prod_{i<j}\left(z_i^{(2)}-z_j^{(2)}\right)^3\prod_{i,j}\left(z_i^{(1)}-z_j^{(2)}\right)^2\Psi_0. \tag{61}$$

This can be seen as a wavefunction of a system of N particles with spin parallel and another N particles with spin antiparallel to the external magnetic field.

7. Conclusion

We have developed a matrix model to describe bilayered QH systems at the filling factor $\nu=q_iK_{ij}^{-1}q_j$. The basic idea was to use two coupled harmonic oscillators in a similar fashion as done by Susskind and Polychronakos. Our model is a generalization of their model and of course reproduces its basic features by taking the coupling parameter K_{12} to be zero.

Starting from an appropriate action we derived the equations of motion for the different matrix model variables. The corresponding Hamiltonian was obtained as the sum of free and interacting terms. A unitary transformation, more precisely a rotation around an angle φ, led to a factorizing Hamiltonian.

Next, we have constructed the ground states of the system in two different ways. The first was based on the unitary transformation and from the ground state after rotation we have derived that before rotating the system. The second was performed directly in terms of a combination of the matrices of harmonic-oscillator operators of two layers. The obtained vacuum configuration involved three different quantities where one describes the inter-layer interaction.

Subsequently, we have investigated the link between our second wavefunction and two others from the literature. After projecting the vacuum configuration on the complex plane and using the Vandermonde determinant, we have shown how the Yoshioka–MacDonald–Girvin wavefunction with the filling factor $\nu = \frac{2}{m+n}$ can be obtained from our model, in particular that corresponding to the $\nu = \frac{1}{2}$ state. Likewise, we have recovered the unpolarized Halperin wavefunction, especially that for the $\nu = \frac{2}{5}$ state.

The case we have studied is in fact just a particular case of more general FQH states where the fluid droplet is assumed to consist of several coupled branches, say M branches. $M = 1$ is the Laughlin (Susskind–Polychronakos) model, $M = 2$ is the model we have discussed here and $M \geqslant 3$ is the generic case, which can be seen as a straightforward generalization of our case.

Of course some important questions still remain to be answered, e.g. about the fractional charge and statistics of the particles and how to describe them in terms of the proposed model. Another interesting question is related to the link between our model and Calogero and super-Calogero models. We will return to these issues and related matter in future.

We close this section by noting that our model will be investigated in the forthcoming work [26] for the case of a single layer. Basically, we will consider the Laughlin liquids in a confining potential that is not of parabolic type and see how this affects the basic features of the Susskind–Polychronakos model.

Acknowledgments

We are very grateful to P Bouwkgnet for extremely helpful discussions. AJ's work is supported by Deutsche Forschungsgemeinschaft within the Schwerpunkt 'Quantum-Hall-Effekt'.

References

[1] Laughlin R B 1989 *Phys. Rev. Lett.* **50** 1559
[2] Tsui D C, Störmer H L and Gossard A C 1982 *Phys. Rev. Lett.* **48** 1555
Prange R E and Girvin S M (ed) 1990 *The Quantum Hall Effect* (New York: Springer)
[3] Halperin B I 1983 *Helv. Phys. Acta* **56** 75
[4] Yoshioka D, MacDonald A H and Girvin S M 1989 *Phys. Rev.* B **39** 1932
[5] For a recent review, see Read N 2001 *Physica* B **298** 121 (*Preprint* cond-mat/0011338)
[6] Koulakov A A, Fogler M M and Shklovskii B I 1996 *Phys. Rev. Lett.* **76** 499
Fogler M M, Koulakov A A and Shklovskii B I 1996 *Phys. Rev.* B **54** 1853
Moessner R and Chalker J T 1996 *Phys. Rev.* B **54** 5006
[7] Willett R L *et al* 1987 *Phys. Rev. Lett.* **59** 1776
[8] Maksym P *et al* 1989 *High Magnetic Fields in Semiconductor Physics II* ed G Landwehr (Berlin: Springer)
[9] Clark R G *et al* 1989 *Phys. Rev. Lett.* **62** 1536
Eisenstein J P *et al* 1989 *Phys. Rev. Lett.* **62** 1540
[10] Das Sarma S and Pinczuk A (ed) 1997 *Perspectives in Quantum Hall Effects* (New York: Wiley)
[11] Suen Y W *et al* 1992 *Phys. Rev. Lett.* **68** 1379
Eisenstein J P *et al* 1992 *Phys. Rev. Lett.* **68** 1383
[12] Lilly M P *et al* 1999 *Phys. Rev. Lett.* **82** 394
Du R R *et al* 1999 *Solid State Commun.* **109** 389

[13] Susskind L 2001 The quantum hall fluid and non-commutative Chern Simons theory *Preprint* hep-th/0101029
[14] Polychronakos A P 2001 *J. High Energy Phys.* JHEP04(2001)011 (*Preprint* hep-th/0103013)
[15] Calogero F 1971 *J. Math. Phys.* **12** 419
[16] Hellerman S and Van Raamsdonk M 2001 *J. High Energy Phys.* JHEP10(2001)039 (*Preprint* hep-th/0103179)
[17] Hellerman S and Susskind L 2001 Realizing the quantum Hall system in string theory *Preprint* hep-th/0107200
[18] Karabali D and Sakita B 2001 *Phys. Rev.* B **64** 245316 (*Preprint* hep-th/0106016)
Karabali D and Sakita B 2002 *Phys. Rev.* B **65** 075304 (*Preprint* hep-th/0107168)
[19] Kobashi Y, Mandal B P and Sugamoto A 2002 Exciton in matrix formulation of quantum Hall effect *Preprint* hep-th/0202050
[20] El Rhalami A, Sahraoui E M and Saidi E H 2002 *J. High Energy Phys.* JHEP05(2002)004 (*Preprint* hep-th/0108096)
Rhalami A El, Sahraoui E M and Saidi E H 2002 *J. High Energy Phys.* JHEP10(2002)39 (*Preprint* hep-th/0208144)
[21] Haldane F D M 1983 *Phys. Rev. Lett.* **51** 605
Halperin B I 1984 *Phys. Rev. Lett.* **52** 1583
[22] Jellal A, Saidi E H and Geyer H B 2002 A matrix model for $\nu_{k_1k_2} = \frac{k_1+k_2}{k_1k_2}$ fractional quantum Hall states *Preprint* hep-th/0204248
[23] Jellal A, Saidi E H, Geyer H B and Römer R A 2003 *J. Phys. Soc. Japan* **72** A 127 (*Preprint* hep-th/0303143)
[24] Wen X G and Zee A 1992 *Phys. Rev.* B **46** 2290
[25] Han D, Kim Y S and Noz M E 1995 *J. Math. Phys.* **36** 3940
[26] Jellal A, Römer R A and Schreiber M in preparation